

GAUSSIAN INTEGRAL MEANS OF ENTIRE FUNCTIONS: LOGARITHMIC CONVEXITY AND CONCAVITY

CHUNJIE WANG AND JIE XIAO

ABSTRACT. For $0 < p < \infty$ and $\alpha \in (-\infty, \infty)$ we determine when the L^p integral mean on $\{z \in \mathbb{C} : |z| \leq r\}$ of an entire function with respect to the Gaussian area measure $e^{-\alpha|z|^2} dA(z)$ is logarithmic convex or logarithmic concave.

1. INTRODUCTION

Let dA be the Euclidean area measure in the finite complex plane \mathbb{C} . For any real number α and $0 < p < \infty$, the Gaussian integral means of an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ are defined by

$$M_{p,\alpha}(f, r) = \frac{\int_{\{z \in \mathbb{C} : |z| \leq r\}} |f(z)|^p e^{-\alpha|z|^2} dA(z)}{\int_{\{z \in \mathbb{C} : |z| \leq r\}} e^{-\alpha|z|^2} dA(z)}, \quad 0 < r < \infty.$$

This concept lies in the theory of Fock spaces; see [7].

The famous Hadamard's three circles theorem for the above entire function f (cf. [1]) states that if

$$\begin{cases} 0 < r_1 < r < r_2 < \infty; \\ M(f, s) = \max\{|f(z)|; |z| \leq s\} \quad \text{for } s \in (0, \infty), \end{cases}$$

then

$$\left(\ln \frac{r_2}{r_1} \right) \ln M(f, r) \leq \left(\ln \frac{r_2}{r} \right) \ln M(f, r_1) + \left(\ln \frac{r}{r_1} \right) \ln M(f, r_2),$$

i.e., $\ln M(f, r)$ is convex in $\ln r$. Continuing from [2] and its prior work [3, 4, 5, 6], this paper investigates such an analogous problem: *When is the function $r \mapsto \ln M_{p,\alpha}(f, r)$ convex or concave in $\ln r$?* In what follows, we will see that a resolution of this question depends on the parameter α and its induced function

$$\varphi(x) = \frac{1 - e^{-\alpha x}}{\alpha}, \quad 0 < x < \infty.$$

Theorem 1. *Let $\alpha < 0$. Suppose both $x \mapsto M(x)$ and $x \mapsto M'(x)$ are positive on $(0, \infty)$. Then the function*

$$x \mapsto \ln \frac{\int_0^x M(t) e^{-\alpha t} dt}{\int_0^x e^{-\alpha t} dt}$$

is convex in $\ln x$ for x in an open interval $I \subset (0, \infty)$ provided that the following conditions are satisfied:

- (i) $x \mapsto \ln M(x)$ is convex in $\ln x$ for $x \in I$;

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(ii)

$$x \frac{M'(x)}{M(x)} \geq \frac{[x(1 - \alpha x) - \varphi(x)][\alpha \varphi(x)^2 - 2(1 + \alpha x)\varphi(x) + 2x]}{(\varphi(x) - x)[x - (1 + \alpha x)\varphi(x)]} \text{ for } x \in I.$$

As a straightforward consequence of Theorem 1, the following logarithmic convexity for $M_{p,\alpha}(f, \cdot)$ is similar to Corollary 8 in [2].

Corollary 2. *Let $(\alpha, p) \in (-\infty, 0) \times (0, \infty)$. If $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function, then $r \mapsto \ln M_{p,\alpha}(f, r)$ is convex in $\ln r$ for $r \in (0, \sqrt{\frac{t_0}{-\alpha}})$, where $t_0 = 1.79 \cdots$ is the unique root of $u(t) = e^t - 1 - t - t^2$ on $(0, \infty)$.*

During the process of extending Theorem 1 from I to $(0, \infty)$, we find the following assertion.

Theorem 3. *Let $\alpha < 0$. Suppose both $x \mapsto M(x)$ and $x \mapsto M'(x)$, are positive on $(0, \infty)$. Then the function*

$$x \mapsto \ln \frac{\int_0^x M(t) e^{-\alpha t} dt}{\int_0^x e^{-\alpha t} dt}$$

is convex in $\ln x$ for $x \in (0, \infty)$ provided that

$$\left(x \frac{M'(x)}{M(x)} \right)' \geq \begin{cases} 0, & x \in (0, x_0); \\ \frac{[x(1 - \alpha x) - \varphi(x)]^2}{4x(\varphi(x) - x)^2}, & x \in [x_0, \infty), \end{cases}$$

where $x_0 = -t_0/\alpha$ is the unique root of $x(1 - \alpha x) - \varphi(x)$ on $(0, \infty)$.

As a by-product of Theorem 3, the following corollary extends the logarithmic convexity of $M_{p,\alpha}(f, \cdot)$ from I to $(0, \infty)$.

Corollary 4. *Let $(\alpha, p) \in (-\infty, 0) \times (0, \infty)$. If $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function, then $r \mapsto \ln M_{p,\alpha}(f, r)$ is convex in $\ln r$ for $r \in (0, \infty)$ if*

$$\left(x \frac{M'(x)}{M(x)} \right)' \geq \frac{[x(1 - \alpha x) - \varphi(x)]^2}{4x[\varphi(x) - x]^2}, \quad x \geq x_0,$$

where $x = r^2$, $M(x) = \int_0^{2\pi} |f(\sqrt{x}e^{i\theta})|^p d\theta$ and x_0 is as the same as above.

However, whenever handling the logarithmic concavity we have only one situation as follows.

Theorem 5. *Let $\alpha \geq 0$. Suppose $M(x)$ & $M'(x)$ are positive and $M''(x)$ exists for $x \in (0, \infty)$. If $x \mapsto \ln M(x)$ is concave in $\ln x$ for $x \in (0, \infty)$, then the function*

$$x \mapsto \ln \frac{\int_0^x M(t) e^{-\alpha t} dt}{\int_0^x e^{-\alpha t} dt}$$

is also concave in $\ln x$ for $x \in (0, \infty)$.

Note that for any nonnegative integer k the classical integral mean of z^k is both logarithmic convex and logarithmic concave. So, we obtain the following corollary, which is part (i) of [2, Theorem 7].

Corollary 6. *Let $(\alpha, p) \in [0, \infty) \times (0, \infty)$. If k is a nonnegative integer, then the function $r \mapsto \ln M_{p,\alpha}(z^k, r)$ is concave in $\ln r$ for $r \in (0, \infty)$.*

Notation. In the forthcoming sections, we will employ the symbol \equiv when a new notation is introduced, but also use the notation $U \sim V$ when U and V have the same sign.

2. FOUR LEMMAS

This section collects four lemmas which will be used in the proofs of Theorems 1-3-5.

The first two lemmas come from [2]

Lemma 7. *Suppose f is positive and twice differentiable on $(0, \infty)$. Then*

- (a) *$f(x)$ is convex in $\ln x$ if and only if $f(x^2)$ is convex in $\ln x$ and $f(x)$ is concave in $\ln x$ if and only if $f(x^2)$ is concave in $\ln x$.*
- (b) *Let*

$$D(f(x)) \equiv \frac{f'(x)}{f(x)} + x \frac{f''(x)}{f(x)} - x \left(\frac{f'(x)}{f(x)} \right)^2.$$

Then $\ln f(x)$ is convex in $\ln x$ if and only if $D(f(x)) \geq 0$ and $\ln f(x)$ is concave in $\ln x$ if and only if $D(f(x)) \leq 0$ for all $x \in (0, \infty)$.

Lemma 8. *Suppose $f = f_1/f_2$ is a quotient of two positive and twice differentiable functions on $(0, \infty)$. Then*

$$D(f(x)) = D(f_1(x)) - D(f_2(x))$$

for $x \in (0, \infty)$. Consequently, $\ln f(x)$ is convex in $\ln x$ if and only if

$$D(f_1(x)) - D(f_2(x)) \geq 0$$

on $(0, \infty)$ and $\ln f(x)$ is concave in $\ln x$ if and only if

$$D(f_1(x)) - D(f_2(x)) \leq 0$$

on $(0, \infty)$.

We next establish several estimates for the function φ .

Lemma 9. *Suppose*

$$\begin{cases} \alpha \in \mathbb{R}; \\ x \in [0, \infty); \\ \varphi = \varphi(x) \equiv \int_0^x e^{-\alpha t^2} 2t dt = \int_0^x e^{-\alpha t} dt = \frac{1}{-\alpha} (e^{-\alpha x} - 1). \end{cases}$$

Then

- (a) $1 - \alpha\varphi(x) = \varphi'(x)$.
- (b) $\varphi(x) - x \geq 0$ when $\alpha \leq 0$ and $\varphi(x) - x \leq 0$ when $\alpha \geq 0$.
- (c) $g_1(x) \equiv x(1 - \alpha x) - \varphi(x) \leq 0$ when $\alpha \geq 0$.
- (d) $g_2(x) \equiv \alpha\varphi^2(x) - 2(1 + \alpha x)\varphi(x) + 2x \leq 0$.
- (e) $g_3(x) \equiv x - (1 + \alpha x)\varphi(x)$ is nonnegative when $\alpha \leq 0$ and not positive when $\alpha \geq 0$.

Proof. Part (a) follows from the fact that

$$\varphi(x) = \frac{1}{-\alpha} (e^{-\alpha x} - 1), \quad \varphi'(x) = e^{-\alpha x}.$$

Part (b) follows from the fact that $e^{-\alpha x} \geq 1$ for $\alpha \leq 0$ and $x \in [0, \infty)$.

A direct computation shows that

$$g_1'(x) = 1 - 2\alpha x - \varphi'(x)$$

and

$$g_1''(x) = -2\alpha - \varphi''(x) = \alpha(\varphi'(x) - 2).$$

It follows that $g_1''(x) \leq 0$ when $\alpha \geq 0$. So we have $g_1'(x) \leq g_1'(0) = 0$ and $g_1(x) \leq g_1(0) = 0$ for all $x \in [0, \infty)$. This proves (c).

Another computation gives

$$\begin{aligned} g_2'(x) &= 2\alpha\varphi\varphi' - 2\alpha\varphi - 2(1 + \alpha x)\varphi' + 2 \\ &= 2\alpha\varphi\varphi' - 2(1 + \alpha x)\varphi' + 2\varphi' \\ &= 2\alpha(\varphi - x)\varphi'. \end{aligned}$$

By part (b), we have $g_2'(x) \leq 0$ for all $x \in [0, \infty)$. Therefore, $g_2(x) \leq g_2(0) = 0$ for all $x \in [0, \infty)$. This proves (d).

A similar computation produces

$$\begin{aligned} g_3'(x) &= 1 - \alpha\varphi(x) - (1 + \alpha x)\varphi'(x) \\ &= -\alpha x\varphi'(x), \end{aligned}$$

which yields $g_3(x) \geq g_3(0) = 0$ for all $x \in [0, \infty)$ when $\alpha \leq 0$ and $g_3(x) \leq g_3(0) = 0$ for all $x \in [0, \infty)$ when $\alpha \geq 0$. This proves (e) and completes the proof of the lemma. \square

Finally, Lemma 9 is applied to derive the following fundamental property.

Lemma 10. *Given a nonconstant entire function $f : \mathbb{C} \rightarrow \mathbb{C}$, suppose*

$$\begin{cases} \alpha \in \mathbb{R}; \\ x \in [0, \infty); \\ p \in (0, \infty); \\ M(x) \equiv M_p(f, \sqrt{x}) = \int_0^{2\pi} |f(\sqrt{x}e^{i\theta})|^p d\theta; \\ h = h(x) \equiv \int_0^x M_p(f, t)e^{-\alpha t^2} 2tdt = \int_0^x M(t)e^{-\alpha t} dt. \end{cases}$$

Let

$$\begin{cases} A = A(x) \equiv \frac{\varphi(x)-x}{\varphi^2(x)}; \\ B = B(x) \equiv (1 - \alpha x) + x \frac{M'(x)}{M(x)}; \\ C = C(x) \equiv x\varphi'(x); \\ \Delta(x) \equiv D(h(x)) - D(\varphi(x)). \end{cases}$$

Then

- (a) $S = S(x) = \sqrt{B^2 - 4AC} > 0 \forall x \in (0, \infty)$.
- (b) $\Delta(x) \sim -A \frac{h^2}{M^2} + B \frac{h}{M} - C \forall x \in (0, \infty)$

Proof. (a) Noticing that $M' > 0$ and $M > 0$, together with $x - (1 + \alpha x)\varphi \geq 0$ by Lemma 9, we have

$$\begin{aligned} B^2 - 4AC &= \left[(1 - \alpha x) + \frac{xM'}{M} \right]^2 - 4x\varphi' \frac{\varphi - x}{\varphi^2} \\ &> (1 - \alpha x)^2 - 4x\varphi' \frac{\varphi - x}{\varphi^2} \\ &= \frac{(2x - (1 + \alpha x)\varphi)^2}{\varphi^2} \\ &> 0. \end{aligned}$$

(b) Since

$$\varphi' = e^{-\alpha x}, \quad \varphi'' = -\alpha e^{-\alpha x},$$

we have

$$D(\varphi(x)) = \frac{(1 - \alpha x)\varphi'}{\varphi} - x \frac{(\varphi')^2}{\varphi^2} = \frac{(\varphi - x)\varphi'}{\varphi^2}.$$

On the other hand,

$$h' = h'(x) = M(x)\varphi',$$

and

$$h'' = h''(x) = [M'(x) - \alpha M(x)]\varphi'.$$

It follows from simple calculations that

$$D(h) = \frac{hh' + xhh'' - x(h')^2}{h^2} = \frac{(1 - \alpha x)Mh + xM'h - xM^2\varphi'}{h^2}\varphi'.$$

Therefore,

$$\begin{aligned}\Delta(x) &= \frac{\varphi'M}{h^2}(hB - CM) - \varphi'A \\ &\sim -A\frac{h^2}{M^2} + B\frac{h}{M} - C.\end{aligned}$$

□

3. PROOFS OF THEOREMS 1&3

To verify Theorems 1&3, we use (ii) of Lemma 7 to show the logarithmic convexity of $h(x)/\varphi(x)$ on $(0, \infty)$. According to Lemma 8, this will be accomplished if we can prove $\Delta(x) \geq 0$.

Suppose that $\alpha < 0$. From Lemma 9 it follows that $A(x)$, $B(x)$ and $C(x)$ are all positive on $(0, \infty)$ as $\alpha \leq 0$ and $M'/M > 0$. By some direct computations, we have

$$\begin{aligned}xA'(x) &= \frac{x}{\varphi^3} [\alpha\varphi^2 - 2(1 + \alpha x)\varphi + 2x], \\ B' &= -\alpha + \left(x \frac{M'(x)}{M(x)}\right)', \\ xC' &= x(1 - \alpha x)\varphi' = (1 - \alpha x)C.\end{aligned}$$

Thus, an application of Lemma 10 yields that $\Delta(x) \geq 0$ is equivalent to

$$(1) \quad -\frac{\sqrt{B^2 - 4AC}}{2A} \leq \frac{h}{M} - \frac{B}{2A} \leq \frac{\sqrt{B^2 - 4AC}}{2A}.$$

Since the function M is positive and increasing, we have

$$B(x) \geq 1 - \alpha x \geq 0, \quad h(x) \leq \int_0^x M(t)\varphi'(t)dt = M(x)\varphi(x).$$

It follows from this, the proof of Lemma 10, part (b) of Lemma 9, and the triangle inequality that

$$\begin{aligned}\frac{B + \sqrt{B^2 - 4AC}}{2A} &\geq \frac{(1 - \alpha x) + \left|1 + \alpha x - \frac{2x}{\varphi}\right|}{2A} \\ &\geq \frac{1 - \alpha x + 1 + \alpha x - \frac{2x}{\varphi}}{2A} \\ &= \frac{2(\varphi - x)}{2A\varphi} = \varphi \geq \frac{h}{M}.\end{aligned}$$

This proves the right half of (1).

To prove the left half of (1), we write

$$(2) \quad \delta = \delta(x) = h - M \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

for $x \in (0, \infty)$ and proceed to show that $\delta(x)$ is nonnegative. It follows from the elementary identity

$$\frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{2C}{B + \sqrt{B^2 - 4AC}}$$

that $\delta(x) \rightarrow 0$ as $x \rightarrow 0^+$. If we can show that $\delta'(x) \geq 0$ for all $x \in (0, \infty)$, then we will obtain

$$\delta(x) \geq \lim_{t \rightarrow 0^+} \delta(t) = 0, \quad x \in (0, \infty).$$

The rest of the proof is thus devoted to proving the inequality $\delta'(x) \geq 0$ for $x \in (0, \infty)$.

By a direct computation, we have

$$\begin{aligned} \delta'(x) &= M\varphi' - \frac{M'A - MA'}{2A^2} \left(B - \sqrt{B^2 - 4AC} \right) \\ &\quad - \frac{M}{2A} \left(B' - \frac{BB' - 2(A'C + AC')}{\sqrt{B^2 - 4AC}} \right) \\ &= \frac{M}{x} \left[C - \left(\frac{xM'}{M} - \frac{x A'}{A} \right) \frac{B - S}{2A} \right. \\ &\quad \left. + \frac{x B'}{2AS} (B - S) - \frac{x A' C + (1 - \alpha x) AC}{AS} \right]. \end{aligned}$$

Noticing that $B + S \geq 0$ for any $\alpha \in (-\infty, \infty)$. Multiplying $\frac{xS(B+S)}{MC}$ on both sides of the above expressions of $\delta'(x)$, and then using (1) and (2), we obtain that

$$\begin{aligned} \delta' &\sim (B + S)S - 2S \left(\frac{xM'}{M} - \frac{x A'}{A} \right) \\ &\quad + 2xB' - \left(\frac{x A'}{A} + 1 - \alpha x \right) (B + S) \\ &= \left(\frac{xM'}{M} - \frac{x A'}{A} \right) (B - S) + 2xB' - 4AC \\ &= \left(\frac{xM'}{M} - \frac{x A'}{A} \right) (B - S) + 2xA'\varphi + 2xD(M(x)) \equiv d_1. \end{aligned}$$

We will determine the sign of d_1 . To this end, we let

$$y = \frac{xM'}{M}$$

and

$$d_2 = \left(y - \frac{x A'}{A} \right) (B - S) + 2xA'\varphi.$$

Note that A, C, A' are independent of y and $B = 1 - \alpha x + y$. A simple computation shows that $S'(y) = B/S$. Multiplying

$$\frac{B + S}{-2xA'\varphi}$$

on the both sides of the above expressions of d_2 , we obtain that

$$\begin{aligned} d_2 &\sim \left(y - \frac{x A'}{A} \right) \cdot \frac{4AC}{-2xA'\varphi} - (B + S) \\ &= \frac{\alpha}{\varphi A'} y + \left(\frac{2x}{\varphi} - 1 - \alpha x \right) - S \equiv d_3. \end{aligned}$$

Proof of Theorem 1, Continued. Since $\alpha < 0$, using the assumption $D(M(x)) \geq 0$ we have $d_1 \geq d_2$. By using Lemma 9(e) we can easily see that

$$\frac{\alpha}{\varphi A'} \geq 1.$$

It follows from a direct computation and Lemma 9 that

$$\begin{aligned} d_2 &\sim \left[\frac{\alpha}{\varphi A'} y + \left(\frac{2x}{\varphi} - 1 - \alpha x \right) \right]^2 - S^2 \\ &= y \left[\left(\frac{\alpha^2}{(\varphi A')^2} - 1 \right) y + 2 \frac{\alpha}{\varphi A'} \left(\frac{2x}{\varphi} - 1 - \alpha x \right) - 2(1 - \alpha x) \right] \\ &\sim y - y_0, \end{aligned}$$

where

$$y_0 = \frac{[x(1 - \alpha x) - \varphi][\alpha \varphi^2 - 2(1 + \alpha x)\varphi + 2x]}{(\varphi - x)[x - (1 + \alpha x)\varphi]}.$$

Since

$$-\alpha [\varphi - x(1 - \alpha x)] = e^{-\alpha x} - 1 + \alpha x - \alpha^2 x^2,$$

we consider

$$u(t) = e^t - 1 - t - t^2, \quad t > 0.$$

It follows from elementary calculus that $u(t)$ has a unique root $t_0 = 1.79 \dots$ on $(0, \infty)$ and $u(t) < 0$ on $(0, t_0)$ and $u(t) > 0$ on (t_0, ∞) . Hence $\varphi - x(1 - \alpha x)$ has a unique root $x_0 = t_0/(-\alpha)$ on $(0, \infty)$ and $\varphi - x(1 - \alpha x) < 0$ on $(0, x_0)$ and $\varphi - x(1 - \alpha x) > 0$ on (x_0, ∞) .

Note that $y_0 \sim \varphi - x(1 - \alpha x)$. When $x \leq x_0$, $y_0 \leq 0$, so we have $d_2 \geq 0$ and hence $d_1 \geq 0$ on $(0, x_0)$. In particular, the function

$$x \mapsto \ln \frac{\int_0^x M(t) e^{-\alpha t} dt}{\int_0^x e^{-\alpha t} dt}$$

is convex in $\ln x$ for $x \in (0, x_0)$. As for $x \in I \cap (x_0, \infty)$, we have $y_0 \geq 0$, the assumption $y \geq y_0$ when $x \in I$ implies $d_2 \geq 0$ and hence $d_1 \geq 0$ on $I \cap (x_0, \infty)$. This shows that d_1 is always nonnegative and completes the proof of Theorem 1. \square

Proof of Theorem 3, Continued. Since $\alpha < 0$, it follows from Lemma 9 that

$$B - S \sim B^2 - S^2 = 4AC \geq 0 \quad \& \quad \frac{x A'}{A} \leq 0.$$

Hence

$$\begin{aligned} d'_2(y) &= B - S + \left(y - \frac{x A'}{A} \right) \left(1 - \frac{B}{S} \right) \\ &\sim S - \left(y - \frac{x A'}{A} \right) \\ &\sim S^2 - \left(y - \frac{x A'}{A} \right)^2 \\ &= \frac{2(\varphi^2 - x(3 + \alpha x)\varphi + 2x^2)}{\varphi(\varphi - x)} y \\ &\quad - \frac{(\varphi - x(1 - \alpha x))(-(1 + 2\alpha x)\varphi^2 + x(5 + 3\alpha x)\varphi - 4x^2)}{\varphi(\varphi - x)^2}. \end{aligned}$$

It follows from Lemma 9 that

$$\varphi - x \geq 0 \quad \& \quad \varphi^2 - x(3 + \alpha x)\varphi + 2x^2 = (\varphi - x)^2 + x(x - (1 + \alpha x)\varphi) \geq 0,$$

and

$$\begin{aligned}
& -(1 + 2\alpha x)\varphi^2 + x(5 + 3\alpha x)\varphi - 4x^2 \\
& = x(\varphi - x) + (x - (1 + \alpha x)\varphi)(\varphi - x) - x[\alpha\varphi^2 - 2(1 + \alpha x)\varphi + 2x] \\
& \geq 0.
\end{aligned}$$

So we have $d'_2(y) \sim y - y^*$, where

$$y^* = \frac{(\varphi - x(1 - \alpha x))(-(1 + 2\alpha x)\varphi^2 + x(5 + 3\alpha x)\varphi - 4x^2)}{2(\varphi - x)(\varphi^2 - x(3 + \alpha x)\varphi + 2x^2)}.$$

Note that $y^* \sim \varphi - x(1 - \alpha x)$.

When $x \leq x_0$, $y^* \leq 0$, $d'_2(y)$ and hence d_2 is nonnegative. As for $x > x_0$, $y^* \geq 0$, $d_2(y)$ attains its minimum value at $y^* \in (0, \infty)$. A direct computation shows that

$$d_2(y^*) = -\frac{1}{2} \left(1 + \frac{\alpha x^2}{\varphi - x} \right)^2.$$

So we have

$$\begin{aligned}
d_1 & \geq 2xD(M(x)) + d_2(y^*) \\
& \sim D(M(x)) - \frac{1}{4x} \left(1 + \frac{\alpha x^2}{\varphi - x} \right)^2.
\end{aligned}$$

This shows that d_1 is always nonnegative and completes the proof of Theorem 3. \square

4. PROOF OF THEOREM 5

To demonstrate Theorem 5, we indicate how to adapt the proof of Theorem 1 or Theorem 3 above to show that $\Delta(x) \leq 0$.

Suppose $\alpha > 0$. Then $A < 0$ by Lemma 9 and so $\Delta(x) \leq 0$ is equivalent to

$$-\frac{\sqrt{B^2 - 4AC}}{2A} \leq \frac{h}{M} - \frac{B}{2A}.$$

So we need only to prove that $\delta = \delta(x)$ defined in (2) is not positive for all $x \in (0, \infty)$. It is enough for us to prove that $\delta'(x) \leq 0$ since $\delta(0) = 0$. We have proved that $\delta'(x) \sim d_1$. Since $M(x)$ is logarithmic concave, that is, $D(M(x)) \leq 0$, we obtain $d_1 \leq d_2$. But $d_2 \sim d_3$. Recall that

$$d_3 = \frac{\alpha}{\varphi A'} y + \left(\frac{2x}{\varphi} - 1 - \alpha x \right) - S.$$

Noticing that

$$\frac{\alpha}{\varphi A'} \leq 0$$

by Lemma 9, we have

$$d_3 \leq \left(\frac{2x}{\varphi} - 1 - \alpha x \right) - S.$$

By the proof of Lemma 10, we have

$$S \geq \left| \frac{2x}{\varphi} - 1 - \alpha x \right|.$$

Thus we get $d_3 \leq 0$ and hence $d_2 \leq 0$. This shows that $d_1 \leq 0$ and completes the proof of Theorem 5.

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CHUNJIE WANG, DEPARTMENT OF MATHEMATICS, HEBEI UNIVERSITY OF TECHNOLOGY, TIANJIN 300401, CHINA
E-mail address: wcj@hebut.edu.cn

JIE XIAO, DEPARTMENT OF MATHEMATICS AND STATISTICS, MEMORIAL UNIVERSITY, ST. JOHN'S, NL A1C 5S7, CANADA
E-mail address: jxiao@mun.ca